Noise Part 1: Time and Frequency Analysis
Noise Part 2:Transitor-level analysis

Analog ICs

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Outline: Part 1

- Overview
- Time-domain analysis
  - Statistical Characteristics and Amplitude Distribution
  - Average noise power and RMS noise voltage
  - Multi-noise-sources (Correlated and Uncorrelated)
- Frequency-domain analysis
  - Noise Power Spectral Density (PSD)
  - Thermal (Johnson) Noise of a Resistor
  - Noise through a Linear Time-Invariant System
- Case study: n-order low-pass filter

Overview

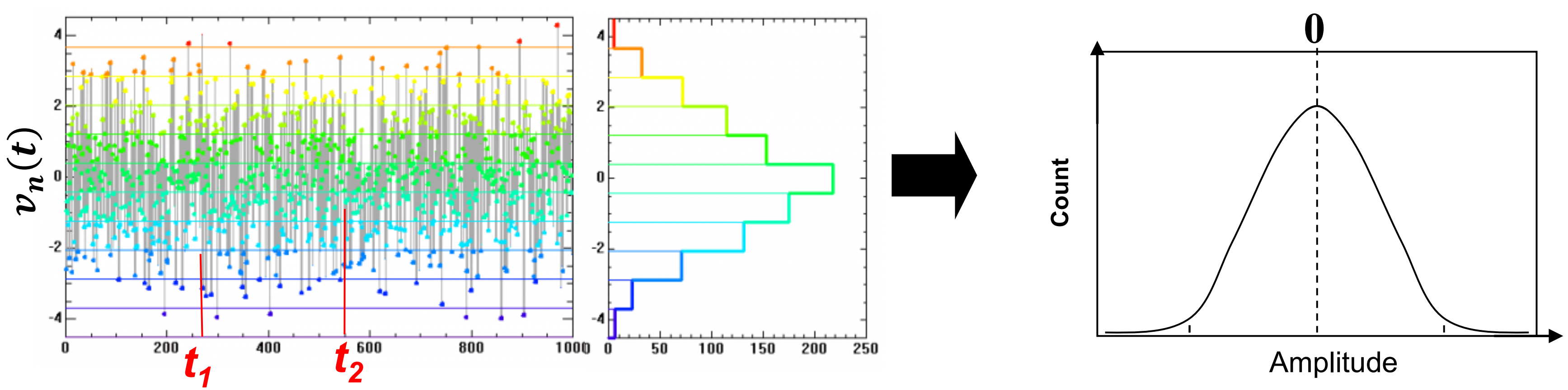
- A signal is always affected by more or less significant random fluctuations called noise.
- Various Origins:
  - Noise inherent to electronic components: e.g., Thermal noise (electrons' agitation) and flicker noise (1/f) due to the defects in the crystal, interface states, ions that trap and release electrons randomly…
  - External noise: electromagnetic (e.g., due to power lines) and Radio frequency, Electrostatic discharge, lightning, atmospheric …
- Limits the sensitivity of any electronic system. (i.e. min. detectable signal)

Time-domain analysis

- Complication : Noise in a non-deterministic signal → The instantaneous value of the noise is impossible to predict (no analytical Formula)

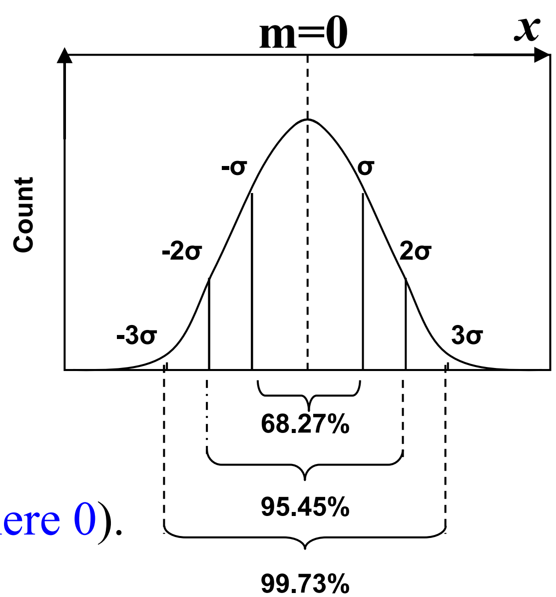
→ Solution: Statistical analysis

- If we observe the noise over a long period, we can construct the amplitude **distribution** (probability curve), indicating how many times each value is reached.



Amplitude Distribution

- **The central limit theorem** states that since noise in electronics arises from a large number of independent random process, it will be described by the normal law (Gaussian) whose appearance and formula are as follows:



$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \frac{-(x-m)^2}{2\sigma^2}$$

- **σ** is the standard deviation. It characterizes the dispersion of the sampled values x (here amplitude) around their mean value **m** (here 0).

- **P(x)** is a probability density → the probability that the noise amplitude is between x_1 and x_2 is given by $\int_{x_1}^{x_2} P(x)dx$

→ How do-we calculate σ?

- It can be demonstrated numerically that the probability for the noise amplitude to be in the range of: **[-σ, +σ] is 68.27%**, and **[-3σ et +3σ] is 99.73%.**

6σ is usually considered as the worse case for *pick to pick* noise *amplitude* in system design

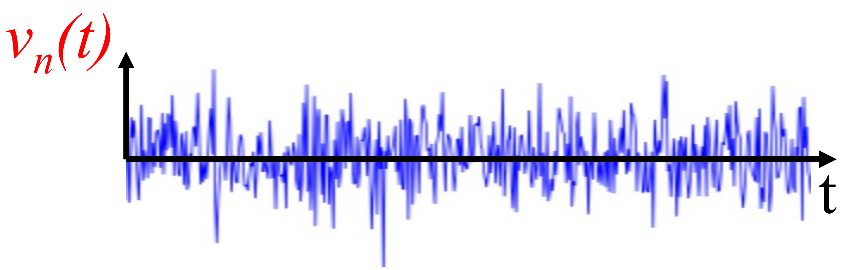
Standard deviation ≡ RMS noise voltage

Since σ_n characterizes the dispersion of the sampled values around the mean m (here $m = \overline{v_n(t)} = 0$) it can be written as:

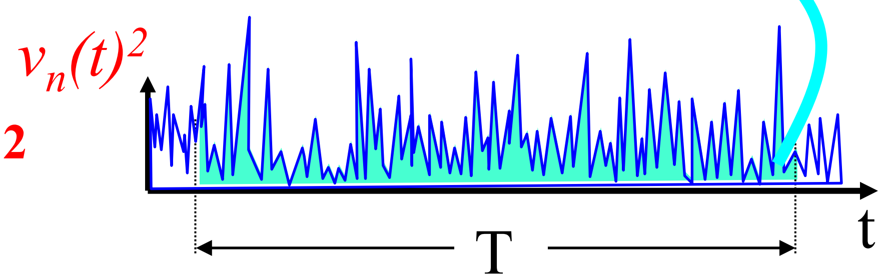
$$\sigma_n = \sqrt{\overline{(v_n(t) - m)^2}} = \sqrt{\overline{\left(v_n(t)\right)^2}} = \sqrt{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v_n(t)^2 dt}$$

$\sigma_n = \sqrt{\overline{v_n{}^2}}$ ≡ **Root Mean Square (RMS)** noise voltage. **[V]**

$\overline{v_n{}^2} = \sigma_n{}^2$ ≡ **Noise power (normalized for R = 1 Ω)**. **[V²]**



$(\ )^2$



Case of several noise sources

- Case of two sources v_{n1} and v_{n2}, the noise power is:

$$\overline{(v_{n1}+v_{n2})^2} = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(v_{n1}(t)+v_{n2}(t)\right)^2 dt$$

$$= \overline{v_{n1}{}^2} + \overline{v_{n2}{}^2} \quad + \quad \underbrace{\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} 2v_{n1}(t)v_{n2}(t)dt}$$

- The residual integral is called the **correlation term.**
- This term → 0 if the sources are not correlated (i.e. independent)
- **In the circuits, the noise sources are in general, uncorrelated.**

→ Therefore: $\overline{\left(v_{n1,2}\right)^2} = \overline{v_{n1}{}^2} + \overline{v_{n2}{}^2}$

→ The superposition holds for the noise powers if the sources are uncorrelated.

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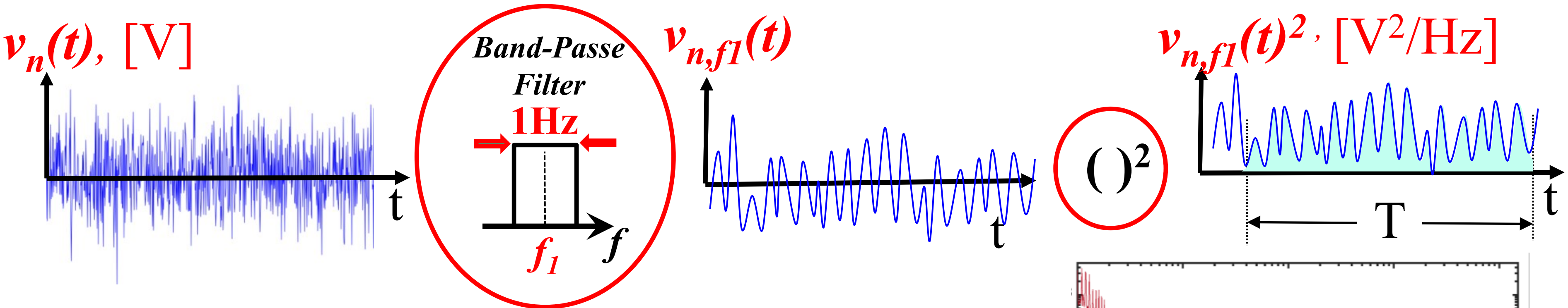
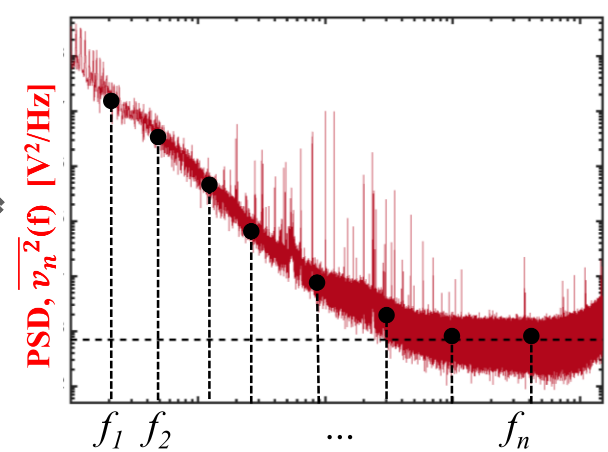
Frequency Noise Spectrum (Power Spectral Density PSD)

- Spectral decomposition of the noise $v_n(t)$ which results in $\overline{v_n^2}(f_1)$ (the average power per Hz) is carried out as follows:



$$\overline{v_n^2}(f_1) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v_{n,f1}(t)^2\,dt$$

[V²/Hz]



➢ The total noise power in [V²] is:

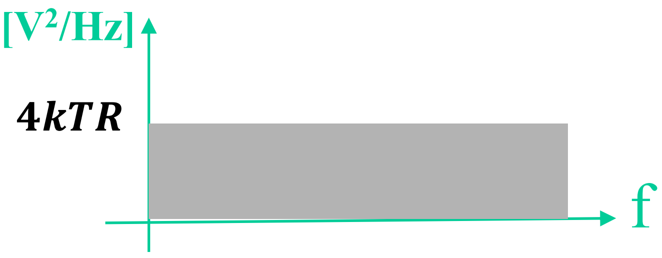
$$\overline{v_n^2} = \int_0^\infty \overline{v_n^2}(f)\,df = \sigma_n^2$$

Example: Thermal (Johnson) Noise of a Resistor

Resistor: Thermal agitation → Random motion of electrons

→ Noise voltage with

PSD: $\overline{v_n^2}(f) = 4kTR$ [V²/Hz]



Where

k is Boltzmann constant = 1,38 10^{-23} [j/K]

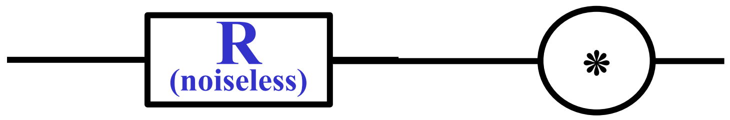
T = Temperature in [°K]

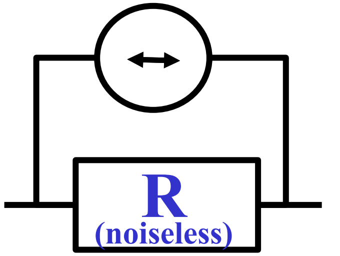
R = resistance in Ω

Norton representation

$\overline{i_n^2}(f) = 4kT/R$ [A²/Hz]

$\overline{v_n^2}(f) = 4kTR$ [V²/Hz]

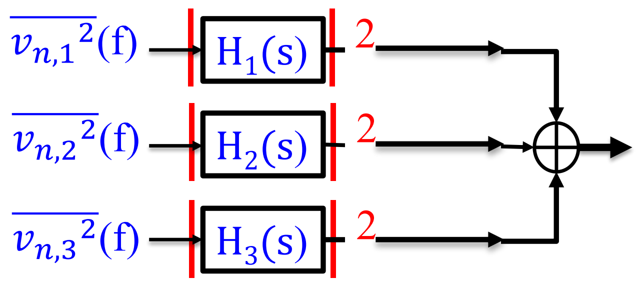




- Particular case: 1kΩ resistor @ 300 °K exhibits an RMS noise voltage of $\sqrt{\overline{v_n^2}} = \sqrt{4kTR} = 4$ nV/$\sqrt{Hz}$.
- → *R = x kΩ results in a noise of* $\sqrt{\overline{v_n^2}} = \sqrt{x}$. 4. nV/ $\sqrt{Hz}$

Noise Through A Linear Time-Invariant System

- Noise propagation & shaping Theorem: A PSD $\overline{v_{n,i}^{2}}(\mathrm{f})$ [V²/Hz] at the input of a linear time-invariant system with a transfer function H(s), gives at the output a PSD $\overline{v_{n,o}^{2}}(\mathrm{f})$ of: $$\overline{v_{n,o}^{2}}(\mathrm{f}) = \overline{v_{n,i}^{2}}(\mathrm{f})\,|H(f)|^{2}\ [\mathrm{V^2/Hz}].$$
  - Case of multiple uncorrelated sources



$$\overline{v_{n,o}^{2}}(\mathrm{f}) = \sum_{j=1}^{3} \overline{v_{n,j}^{2}}(\mathrm{f})\,|H_j(f)|^{2}$$

  - Total output noise power of :

$$\overline{v_{n,o}^{2}} = \int_{0}^{\infty} \overline{v_{n,i}^{2}}(\mathrm{f})\,|H(f)|^{2} df\ \ [\mathrm{V^2}]$$

Signal to noise ratio: SNR

- The signal-to-noise ratio (SNR) is the ratio of the desired signal power to the undesired noise power.

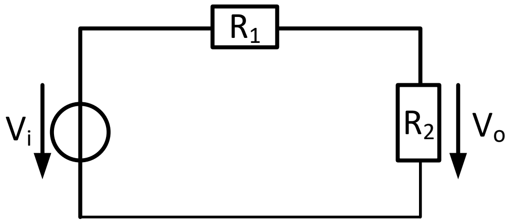
$$SNR_{dB} = \left.\frac{v_{s,RMS}}{v_{n,RMS}}\right|_{dB} = 20\log\left(\frac{v_{s,RMS}}{v_{n,RMS}}\right) = 10\log\left(\frac{\overline{{v_s}^2}}{\overline{{v_n}^2}}\right)$$

- Why is SNR important?:
  - Optimizing noise can degrade the gain and so the desired signal

    → Better optimize SNR.

Example 1 a : Resistive divider

- Use the superposition noise shaping theorem to determine the output noise of a resistive divider (R_1, R_2) and deduce the impact of each resistor on noise performance.



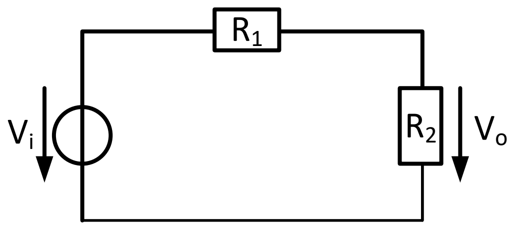
$$\overline{v_{n,o}^2}(f) = 4kT(R_1//R_2)\ [V^2/Hz]$$

$$\overline{v_{n,o}^2} = 4kT(R_1//R_2)\ \Delta f\ [V^2]$$

$\overline{v_{n,o}^2}$ ↘ if R_1 ↘ and R_2 ↘

Example 1 b : Resistive divider

Revaluate the noise performance using the signal-to-noise ratio (SNR).



$\overline{v_{n,o}^{2}}(f) = 4kT(R_1//R_2)\ [V^2/Hz]$

$\overline{v_{n,o}^{2}} = 4kT(R_1//R_2)\ \Delta f\ [V^2]$

$$|\boldsymbol{A_v}| = \frac{R_2}{R_2+R_1}$$

$$SNR = \frac{\overline{v_o^2}}{\overline{v_{n,o}^2}} = \frac{\left(\frac{R_2}{R_2+R_1}\right)^2 \overline{v_i^2}}{4kT\frac{R_2R_1}{R_2+R_1}\Delta f}$$

$$= \frac{1}{\Delta f}\frac{R_2}{4kTR_1(R_2+R_1)}\overline{v_i^2} = \frac{1}{\Delta f}\frac{\overline{v_i^2}}{4kT}\frac{1}{R_1\left(1+\frac{R_1}{R_2}\right)}$$

$\overline{v_{n,o}^{2}}$ ↘ **if** R_1 ↘ **and** R_2 ↘

while SNR ↗ **if** R_1 ↘ **and** R_2 ↗

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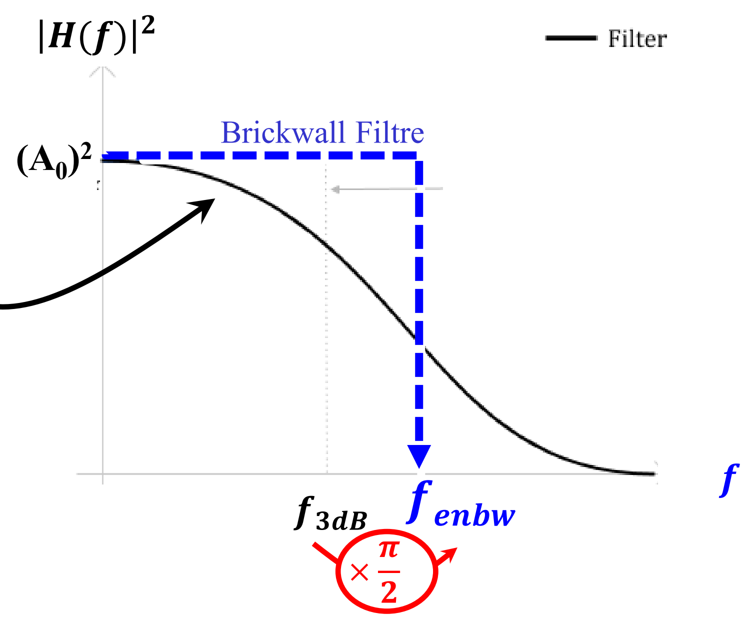
Example 2: First-order low-pass filter

- Example: case of a white noise $(\overline{v_{n,i}^{\,2}}(\mathrm{f}) = N = cst\ [V^2/Hz])$ at the input of a first-order low-pass filter (H(s)) with a cutoff frequency $f_c = f_{3dB}$.
- Use the noise shaping theorem to estimate its output noise power $\overline{v_{n,o}^{\,2}}$ [V^2] and its maximum peak-to-peak noise value $\boldsymbol{v_{n,pp,max}}$ [V]. Note: $\mathrm{H}(s) = \frac{A_0}{1+\frac{jf}{fc}}$ *and* $\int \frac{dx}{1+x^2} = tan^{-1}x + c$

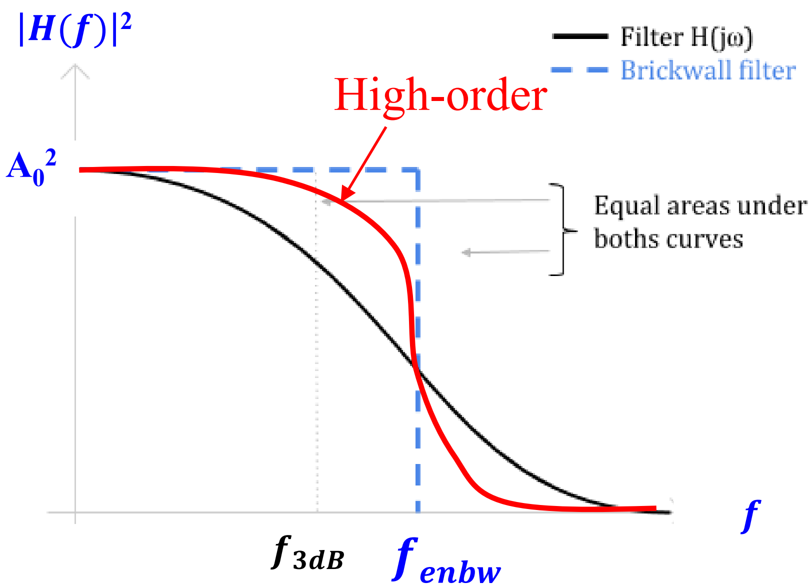
$$\overline{v_{n,o}^{\,2}} = N\int_0^{\infty} \frac{A_0^{\,2}}{1+\left(\frac{f}{f_{3dB}}\right)^2} df \;=\; \mathrm{N}.\boldsymbol{A_0}^{\mathbf{2}}\,\boldsymbol{f_{3dB}}\left[Arctg\ x\right]_0^{\infty} = \mathrm{N}\ \boldsymbol{A_0}^{\mathbf{2}}\ \underbrace{\boldsymbol{f_{3dB}}\,\frac{\boldsymbol{\pi}}{\mathbf{2}}}_{\boldsymbol{f_{ENBW}}}$$

$\boldsymbol{f_{ENBW}}$

Equivalent Noise Bandwidth



Equivalent noise Bandwidth of nth-order filter



$$H(f) = \frac{A_0}{\left(1 + j\frac{f}{f_p}\right)^n}$$

Filter Order n	f_{enbw}/f_{3dB}
1	$\pi/2 = 1.57$
2	1.22
3	1.15
4	1.13

- As the order of the filter increases, the filter roll-off increases and approach that of the brick-wall filter.

$$i.e.\quad \frac{f_{ENBW}}{f_{3dB}} \xrightarrow[n\nearrow]{} 1$$

$$\overline{v_{n,o}{}^2} = N.\,A_0{}^2\, f_{enbw} = \sigma^2$$

Note: $f_{3dB} = f_p\sqrt{2^{\frac{1}{n}} - 1}$

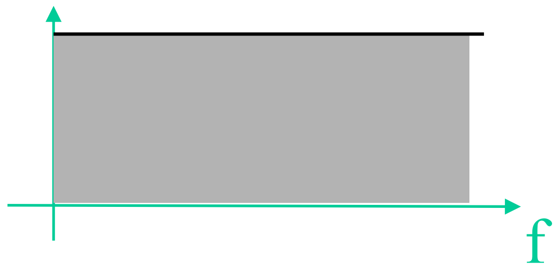
Example 3:

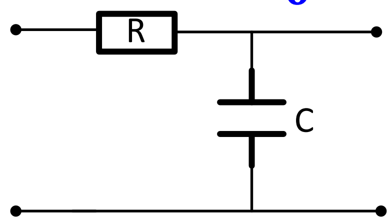
- Demonstrate that the total noise power of a simple low-pass RC circuit is: $\overline{v_n^2} = \dfrac{kT}{C}$
- Explain intuitively why this noise is independent of the value of R:

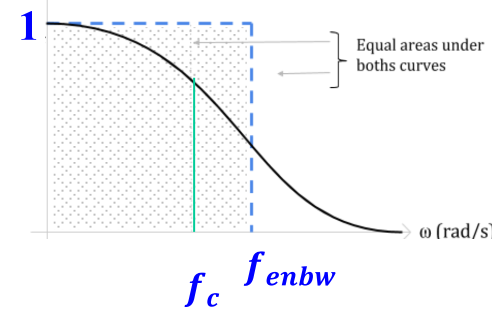
$$\int_0^\infty 4kTR.\,|H(f)|^2 df$$

[V²/Hz]

$4kTR$







$$\overline{v_{n,o}^2} = 4kTRf_{enbw} = 4kTR\frac{\pi}{2}f_c = 4kTR\frac{\pi}{2}\frac{1}{2\pi RC} = \frac{kT}{C}$$

If R ↗ → Noise PSD ↗ but f_c ↘